ESTIMATION OF LOCAL BANDWIDTH FOR NON-STATIONARY SIGNALS

by

SUMEET GOYAL

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

APRIL, 1998

ESTIMATION OF LOCAL BANDWIDTH FOR NON-STATIONARY SIGNALS

A Thesis Submitted

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF TECHNOLOGY

by

Sumeet Goyal

to the

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

INDIA

April 1998

Certificate

It is certified that the work contained in the thesis entitled ESTIMATION OF LOCAL BANDWIDTH FOR NON-STATIONARY SIGNALS, by Sumeet Goyal (Roll No 9610456), has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

Pradip Sircar

April 1998

Dr. P Sircar
Assistant Professor
Department of Electrical Engineering
I.I.T. Kanpur

Abstract

Local bandwidth has been estimated for the non-stationary signals. For estimating the local bandwidth the real time data (signal) has been approximated using orthogonal polynomial series. Assumption has been made that in the bounded region the highest frequency will dominate in the higher order derivatives (say higher than sixth order derivative). Derivatives have been estimated using orthogonal polynomial approximation. Before testing the efficacy of this method on real time signals, it has been tested on sine-waves and mixture of sine-waves of different sets of frequencies. This method of estimation of local bandwidth has been applied to real time non-stationary signals of ECG and EEG. The results have also been cross-verified by auto-regressive method of frequency estimation.

Dedicated To

My Parents and Wife

Acknowledgements

I express my deepest gratitude to my thesis supevisor Dr. P. Sircar for providing me valuable suggestions and encouragement during my M. Tech. project work. I really appreciate him for his kind nature and patience he displayed to discuss the technical matters at any odd hours. Without his help, this work would never have been completed.

I would like to thank all of my classmates for their good company, and in particular to Murty, who helped me to get over difficult times. A special thanks to DD Engineer Jethi for his support, and last but not the least to my wife, Rekha, for her eternal patience and bearing with me during my moods and tantrums.

Contents

List of Figures

Chapter 1

Introduction

To find the local bandwidth implies, finding the highest frequency present in a small span of the signal. This frequency represents the local bandwidth or the highest frequency component associated with the local region of the signal [1].

1.1 Motivation for the Thesis

Finding the local bandwidth is an important part of the signal analysis. To estimate the local bandwidth reasonably accurately and quickly has a number of uses. One of the numerous uses that comes to mind is in non-uniform sampling, which in turn founds applications in such diverse fields as machine vision, radio astronomy and computed tomography. In non-uniform sampling finding the local bandwidth enable us to change the sampling interval accordingly. Reconstruction of the signal from non-uniform samples, taken at a frequency of twice the local bandwidth, is possible was shown by Clark and Palmer [1]. This conclusion has been reached from a different direction by Horiuchi [2]. The conventional methods for calculating the local bandwidth are Short Time Fourier Transform (STFT), Auto-Regressive Frequency Estimation (ARFE) and Maximum Like-lihood Estimation of Frequency (MLE). Though there are a number of other ways to do the frequency analysis also, but most of them are quite computationally intensive. Thus to find an efficient way for finding the local bandwidth is important.

1.2 Work Done in this Area

Initially the previous work of reference [3] was tried to be extended. In estimating the local bandwidth, for calculating the higher order derivatives use of the divided difference table had been made in the previous work. This divided difference approach is relatively a quite simplistic one. A more exhaustive approach to find the higher order derivative was tried in the present work. The function represented in the sampled data was approximated by a polynomial of suitable order. This method had its own limitations, if the frequencies present in the signal were closer together, it required higher order derivative to be evaluated thereby requiring higher order polynomial to be fitted, which became computationally intensive and also did not provide quite satisfactory results due to inherent problem of error propagation in the divided difference table.

1.3 Objective

In the present work an attempt has been made to estimate the local bandwidth of the signals using orthogonal polynomials. A much better way to find the higher order derivatives was to use orthogonal polynomial approximation. In which higher order derivatives can be evaluated by using simple recurrence relations and also the polynomial approximation using minimum error variance criterion has inherent noise rejection capabilities. To see the feasibility of using the orthogonal polynomial to find the higher order derivatives and thereby finding the local bandwidth, it was first used to find the frequency of a single sine wave. Then the maximum frequency for a signal which contained a mixture of sine-waves was taken. Later, this method was tested on real time ECG and EEG signals.

1.4 Organisation of Thesis

This thesis work has been organised into five chapters. Chapter 1 deals with the introduction. Chapter 2 gives basics of orthogonal polynomial approximation which constitutes an important part of this thesis for estimating the frequency. Chapter 3

describes the basic principle of estimating the frequency. Its salient features, limitations and constraints has also been discussed in this chapter. Its application to the real time ECG and EEG signal has also been dealt with in this very chapter. Chapter 4 gives the results and analysis. Chapter 5 concludes the thesis work with suggestions for future work.

Appendix A gives the algorithm for the generation of orthogonal polynomials and their derivatives.

Chapter 2

Orthogonal Polynomial Approximation

The problem of approximating a function whose values at a sequence of points are generally known only empirically, and thus are subject to inherent errors which may be large, is a serious one. The Orthogonal polynomial approximation (using least square approximation) is a technique by which noisy functional values may be used to generate a smooth approximation to the function. This smooth approximation can then be used to approximate the derivative of the function more accurately than exact approximations [5]. The procedure to generate the reccurence relations of the functional approximation and its derivative approximation through orthogonal polynomial approximation has been given in the next section.

2.1 Reccurence Relations for Generating Orth. Poly.

If the abscissas of a function given at m discrete points be (x_i, f_i) then it can be approximated by use of a series of orthogonal polynomials $p_j(x)$.

Suppose the set $\{p_j(x)\}$ is any sequence of polynomials satisfying the orthogonality relationship (2.1) with respect to the sequence of data points $\{x_i\}$ i.e.,

$$\sum_{i=1}^{m} p_j(x_i)p_k(x_i) = \begin{cases} 1 & j = k \\ 0 & j \neq k, \end{cases} \tag{2.1}$$

Then the polynomials are evaluated by the recurrence relation [6]

$$p_{j+1}(x) = (x - a_{j+1})p_j(x) - b_j p_{j-1}(x) \qquad j \geq 0, \tag{2.2}$$

with $p_0(x) = 1$, $p_{-1}(x) = 0$

and a_{j+1} and b_j are constants given by

$$a_{j+1} = \frac{\sum_{i=1}^{m} x_i [p_j(x_i)]^2}{N_j}$$

$$b_j = \frac{N_j}{N_{j-1}}$$

$$N_j = \sum_{i-1}^{m} [p_j(x_i)]^2 \qquad (2.3)$$

where N_j is the scalar product of the orthogonal functions.

The generating polynomials for the first and the n_{th} order derivatives can be evaluated in a similar way by differentiating the equation (2.2) required number of times. i.e.,

$$p^1_{j+1}(x) = p_j(x) + (x - a_{j+1})p^1_j(x) - b_j p^1_{j-1}(x), \qquad j \geq 1, \qquad (2.4)$$

with $p^1_0(x) = 0$, $p^1_1(x) = 1$.

,

$$p^n_{j+1}(x) = np^{n-1}_j(x) + (x - a_{j+1})p^n_j(x) - b_j p^n_{j-1}(x), \qquad n \geq 2, j \geq 1, \qquad (2.5)$$

with $p^n_0(x) = 0$, $p^n_1(x) = 0$

where a_{j+1} and b_j are defined as in equation (2.3).

Then the function given by m points (x_i, f_i) can be approximated by the least-squares approximation (orthogonal polynomial approximation)

$$y_n(x) = \sum_{j=0}^{n} A_j p_j(x) \qquad n \leq m - 1, \qquad (2.6)$$

and the first and second order derivatives are approximated by

$$y^1_n(x) = \sum_{j=1}^{n} A_j p^1_j(x) \qquad n \leq m, \qquad (2.7)$$

and

$$y^2_n(x) = \sum_{j=2}^{n} A_j p^2_j(x) \qquad n \leq m, \qquad (2.8)$$

where,

$$A_j = \frac{\sum_{i=1}^{m} p_j(x_i) f_i}{N_j} \qquad 0 \leq j \leq m$$

The representing polynomial series should be truncated at the polynomial degree of approximation where the error variance given by

$$\sigma_n^2 = \frac{\sum_{i=1}^{m}[\overline{f}_i - \sum_{j=0}^{n} A_j p_j(x)]^2}{m-n-1} \tag{2.9}$$

is either minimum or does not decrease appreciably any further with increase of polynomial degree. It is counter-productive to include more terms because the redundant ones describe the noise in f_i.

Refer appendix A for the algorithm of generation of orthogonal polynomials and their derivatives.

2.2 Graphs

Figure 2.1 shows the graph for orthogonal polynomials of degree $j = 0, ...5$. First and second derivatives of the orthogonal polynomials of degree $j = 0, ...5$ have been given in Figure 2.2 and Figure 2.3 respectively. Similarly graphs for the orthogonal polynomials of degree $j = 9, 10, 11$ and the corresponding sixth order derivatives have been shown in the Figure 2.4 and Figure 2.5 respectively.

2.3 Advantages of Orthogonal Polynomial Approximation

Some of the distinct advantages of orthogonal polynomial approximation are given below [5], [8]:

- The orthogonal polynomial approximation uses the least-squares criterion. By this, the approximated function is not only accurate at data points, but also at other values in a given interval.

- It provides a convenient method of approximating a function since it can be easily implemented on a computer.
- In this technique the minimum error-variance criterion is used to get the polynomial degree of the approximation. This feature ensures maximum noise rejection at the sampled set of data points.
- The differentiation of the polynomial approximation can be easily mechanized on a computer.
- The numerical stability for the orthogonal polynomials is quite good.
- The math analysis required for the orthogonal polynomials is easy though the evaluation count required is a little bit high.

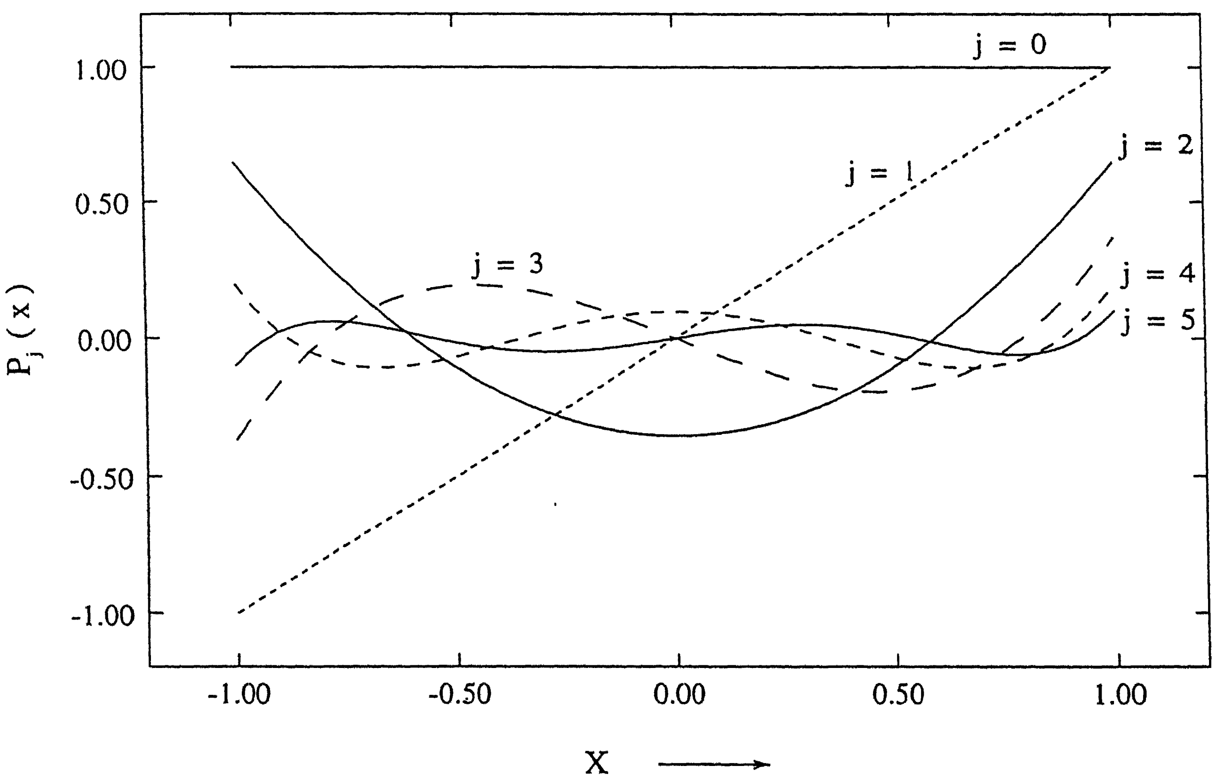

Figure 2.1: Orthogonal Polynomials of Degree $j = 0, 1, ...5$.

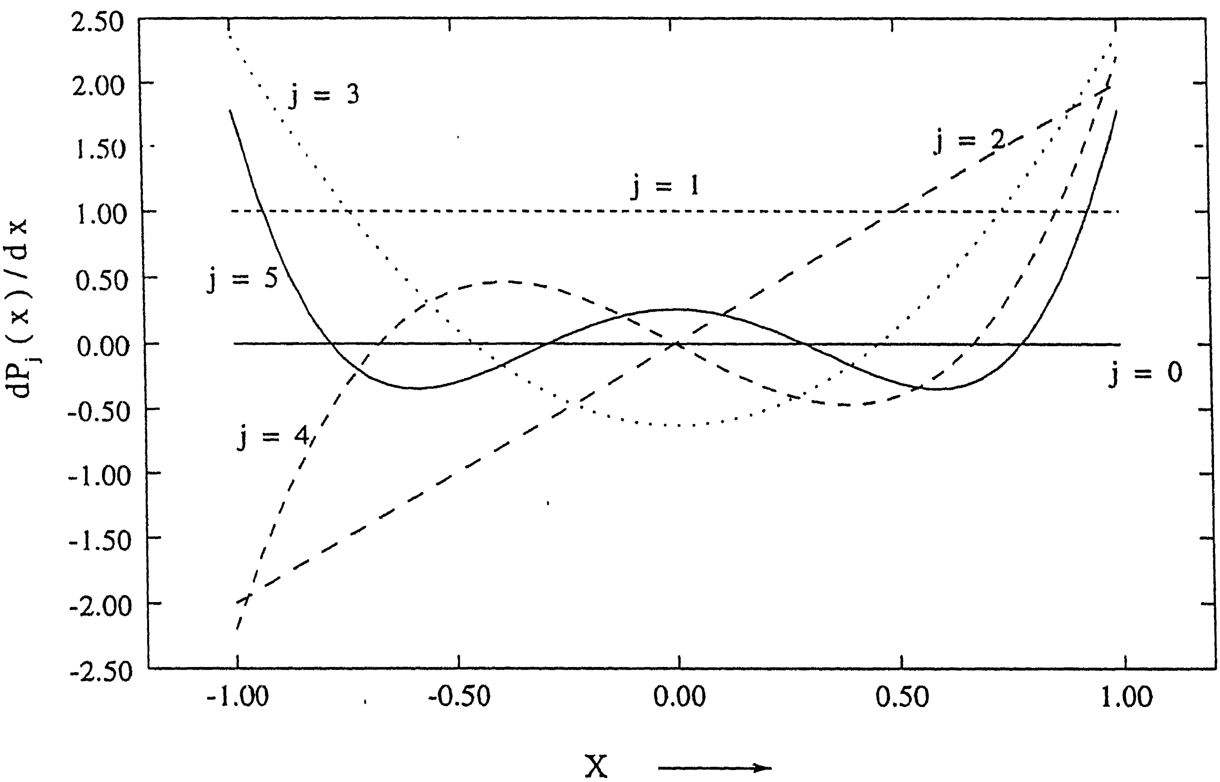

Figure 2.2: First Order Derivative of Orth. Poly. of Deg. $j = 0, 1, ...5$.

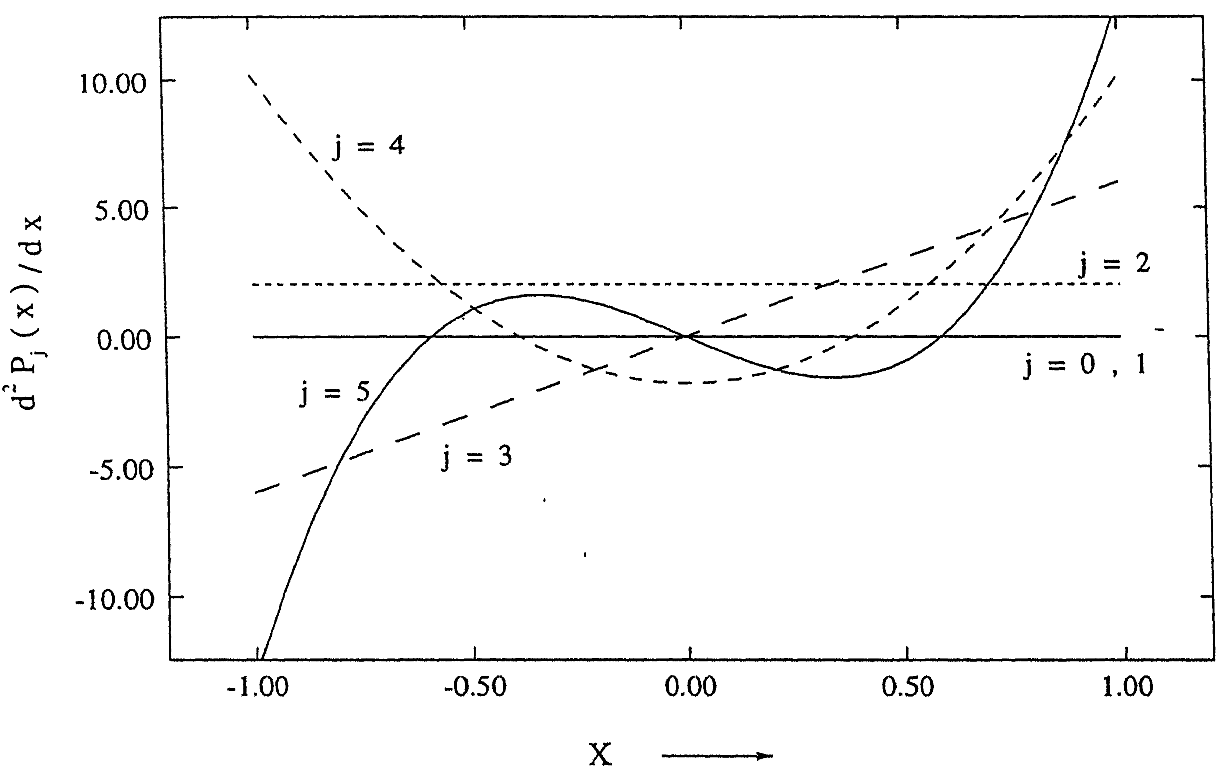

Figure 2.3: Second Order Derivative of Orth. Poly. of Deg. $j = 0, 1, ...5$.

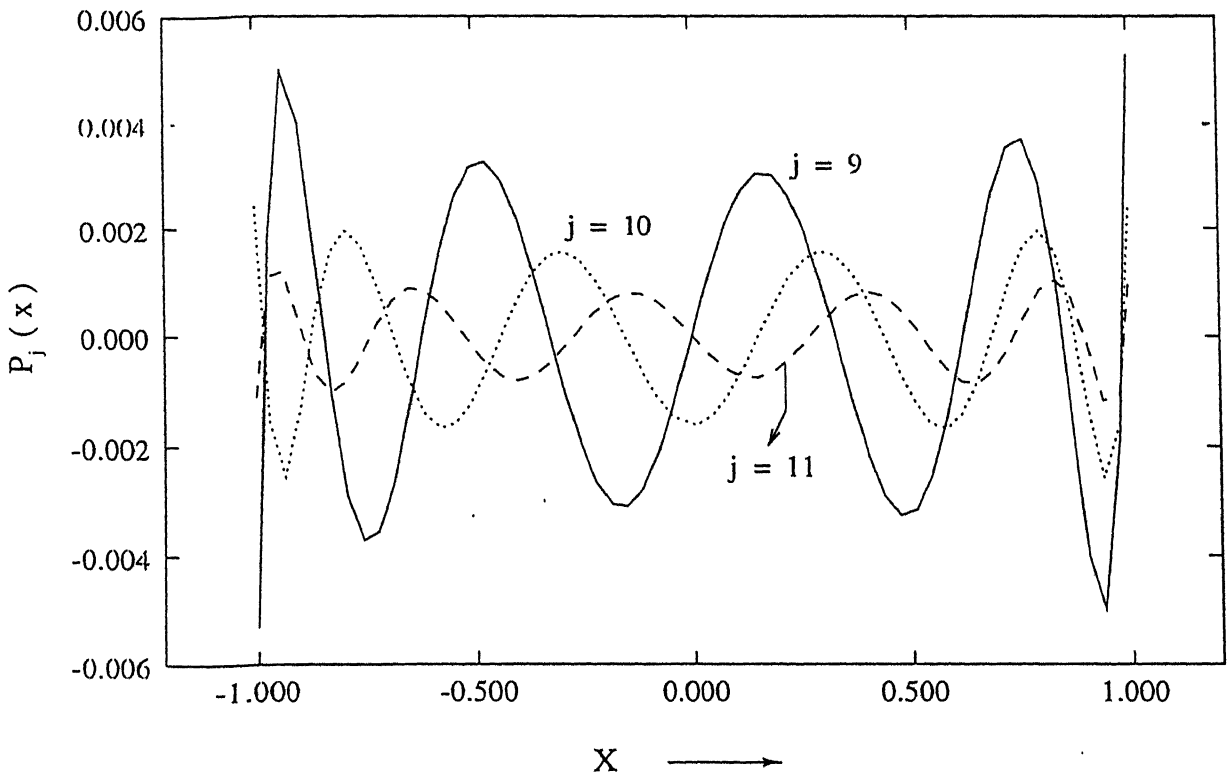

Figure 2.4: Orthogonal Polynomials of Degree $j = 9, 10$ and 11.

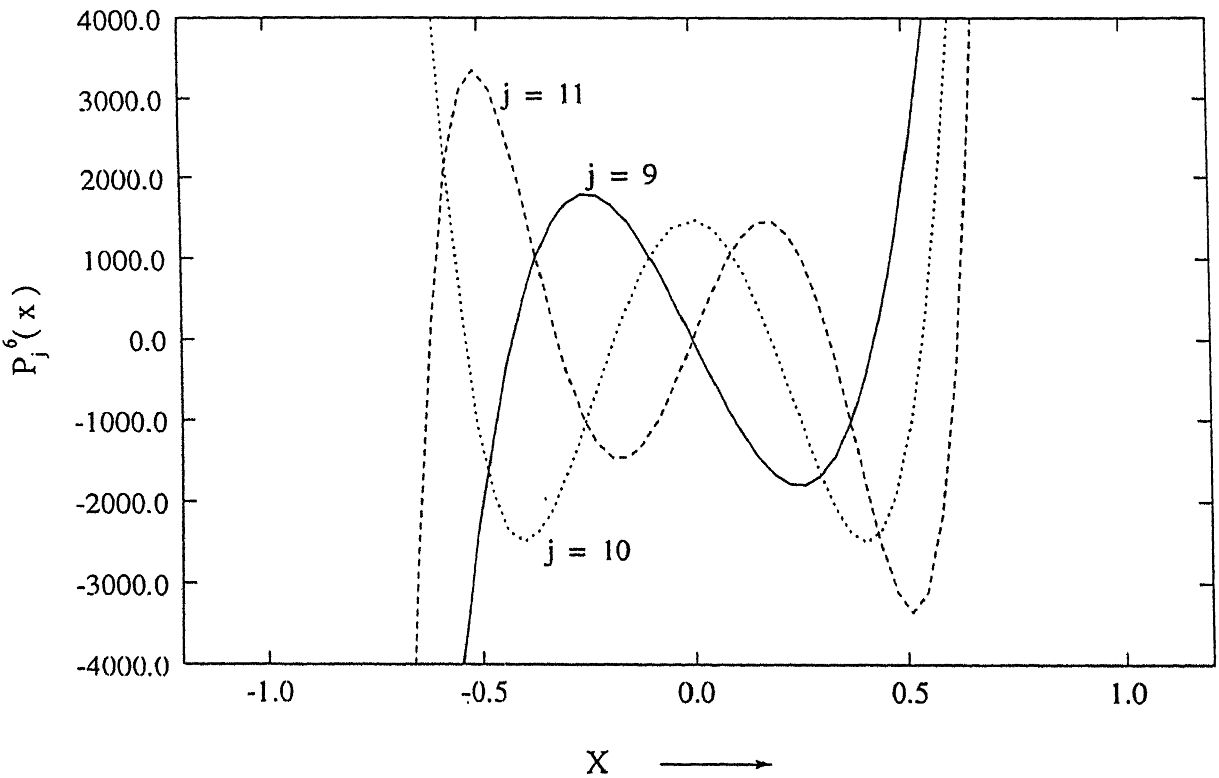

Figure 2.5: Sixth Order Derivative of Orth. Poly. of Deg. $j = 9, 10$ and 11.

Chapter 3

Theory and Implementation

3.1 Basic Principle for Local Bandwidth Estimation

Consider a signal consisting of a mixture of various sinusoids:

$$g(t) = \alpha_1 \sin \omega_1 t + \alpha_2 \sin \omega_2 t + \alpha_3 \sin \omega_3 t \tag{3.1}$$

where α_1, α_2, α_3 are the amplitudes associated with the sinusoids and $\omega(= 2\pi f)$ is the angular frequency and f is the frequency of the sinsoid. Here the assumption has been made that $\omega_1 > \omega_2 > \omega_3$ and the difference in frequencies is such that if we take the higher order derivative then the effect of the lower frequency can be neglected i.e.,

$$g^6(t) = -\omega_1^6 \alpha \sin \omega_1 t - \omega_2^6 \beta \sin \omega_2 t - \omega_3^6 \gamma \sin \omega_3 t \tag{3.2}$$

can be approximated as

$$g^6(t) \cong -\omega_1^6 \alpha \sin \omega_1 t \tag{3.3}$$

i.e., ω_1^6 is sufficiently larger than ω_2^6 and ω_3^6.

Similarly the eighth order derivative of the function $g(t)$ can be approximated as

$$g^8(t) \cong \omega_1^8 \alpha \sin \omega_1 t \tag{3.4}$$

It is assumed that the signal $g(t)$ is bandlimited.

From equation 3.3 and equation 3.4 we get,

$$\omega_1 = \sqrt{\frac{-g^8(t)}{g^6(t)}} \tag{3.5}$$

this implies that,

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{-g^8(t)}{g^6(t)}} \tag{3.6}$$

we can calculate the highest frequency f_1 i.e., the local bandwidth. In this way by calculating the higher order derivative we are able to get the local bandwidth of the signal.

3.1.1 Limitations

- If the frequencies present in the signal are not very much far apart, then to distinguish the highest frequency a much higher derivative may be required to be calculated, and we know that calculating derivatives has inherent errors.
- If two close frequencies are present in the signal then the higher order derivative may simply not be approximated by equation 3.2.

3.2 Theory

In the previous work [3] the derivatives over 10 sample points were calculated by

$$\frac{f^k(t_j)}{k!} = f[t_1, t_2, \cdots\cdots\cdots\cdots t_9] \tag{3.7}$$

where $f[t_1, t_2, \cdots\cdots\cdots\cdots t_n]$ represents the divided difference. Taking derivative in this manner is a rather simplistic approach. But here in this thesis a more elaborate and accurate way to find out the derivatives was tried. A ninth degree polynomial was fitted in the 10 points [9].

$$a_9t^9 + a_8t^8 + \cdots\cdots\cdots\cdots\cdots\cdots a_1t + a_0 = g \tag{3.8}$$

Then the ninth order derivative will be given by

$$\frac{d^9 g}{dt^9} = 362880a_9 \tag{3.9}$$

and the seventh order derivative at any point t will be given by

$$\frac{d^7 g}{dt^7} = 181440a_9 t^2 + 40320a_8 t + 5040a_7 \tag{3.10}$$

Software was written to calculate various order derivatives and to estimate the maximum frequency i.e., the local bandwidth from 10 sampled points. Derivatives were also tried to be found by the differentiation formulae given by W. G. Bickley in a tabular form [7].

$$\frac{h^m}{m!} y_r^m \approx \frac{1}{6!}(A_0 y_0 + A_2 y_2 + \ldots\ldots\ldots\ldots + A_6 y_6) \tag{3.11}$$

where y_r^m stand for the value of $(d^m y/dx^m)_r$, i.e., for the m_{th} differential of y at the point $x/l = r/n$. Then, h denoting the quantity l/n, The values of $A_0, A_1 \ldots\ldots\ldots, A_6$ (for $n = 6$) in equation 3.11, for $m = 1, 2, \ldots\ldots..6$ and for $r = 0, 1, 2, \ldots\ldots..6$, are given in tabular form.

But the results were not very satisfactory and also these methods had some limitations. If higher order derivative than sixth order was required to distinguish the highest frequency then the sampled points required will increase requiring a higher order polynomial to be fitted. It became very computationaly intensive as well as the approximation of the derivative was not very accurate.

This resulted in looking for a better and easier way to represent the higher order derivatives. The answer was found in the orthogonal polynomial approximation.

The orthogonal polynomial approximation with minimum error variance provides a suitable representation when not only the function values, but also the derivative values are to be computed. With orthogonal polynomial approximation, the higher order derivatives can easily be approximated in terms of the derived orthogonal polynomials which are derived by the use of simple recurrence relations. When finding the local bandwidth by finding the higher order derivative through orthogonal polynomials some important things should be observed as explained below [4].

3.2.1 Important Parameters for Orthogonal Polynomial Approximation

Choice of Span

When the signal consists of fast variations alongwith the presence of very slow variations as is the case with the ECG signal. In such cases choosing the number of sample points in the span becomes an important parameter in estimating the local bandwidth of the signal. For a quickly changing signal if we take a large span (i.e., more number of sample points) then the fast variations in the signal will be averaged out and we would not get a correct estimate of the highest frequency present in the local region. Whereas if the span is small in case of the slowly varying signal then again the estimate would not be accurate. When the noise is present alongwith the signal, in such cases it is better to take a larger span of the signal to estimate the frequency. Then that span is approximated by a lower order of approximation (i.e., averaging out the noise) thereby giving the correct estimate of the local bandwidth. An indication towards the frequency contents can be got from the coefficients of the polynomial and the error variances. The use of this tecnique has been made in the case of ECG signal which has both very slow and very fast changes in the signal.

When the fast signal variations are encountered in the ECG signal, the first error variance component becomes very high, and based on this fact the single span i.e., number of sampled points chosen to estimate the frequency is made equal to 22 sample points. When the slow variations are present in the signal, the span is chosen as 72. In this way we are able to get quite accurate estimates of frequencies even for a signal which has wide range of sudden variations in the frequencies.

Choice of Order

Choosing the order of approximation is an important factor in getting good results. Experimentation on a large number of signals give an idea on how to choose the order of approximation. Basically the thumb rule can be established as:

In the presence of noise mixed with the sampled signal values, the series

$$f(t) = \sum_{j=1}^{N} c_j P_{j-1}(t) \tag{3.12}$$

where $P_{j-1}(t)$ is a polynomial of degree $(j-1)$, and c_j is its coefficient, should be truncated at the order of approximation where the error variance

$$\sigma_j^2 = \frac{\sum_{k=1}^{K} [f(t_k) - \sum_{j=1}^{J} c_j P_{j-1}(t_k)]^2}{K-J} \tag{3.13}$$

where $J \leq N \leq K$,

is either minimum or does not decrease appreciably any further with the increase of approximation order.

The error-variance vs order plot for two cases (one on 22 points data taken from ECG samples and the other for 35 points data taken from EEG signal) have been shown in Figure 3.2 and Figure 3.1 . Figure 3.2 shows a simple case in which the order of approximation has been chosen as 10, as is expected. But whereas Figure 3.1 gives a typical case. The order of approximation for this case has been chosen as 6 whereas it could very well have been taken as 26. But that would have given us a wrong estimate as it would be approximating quite a lot of noise in that case.

Limitations [4]

- Since the polynomials are computed by recurrence relation, the quantization error introduced in the higher order polynomials can be a problem when a large number of data points are processed and (or) the additive noise is not purely white. It is then advisable to restrict the value of N i.e., the span and number of sampled data points, so that the region of oscillation in the error variance plot, as shown in the Figure 3.1 can be avoided.

- When noise is present, the errors of approximation in the derivatives may become large at the ends of the sampling span. Hence the result can be expected to be in error at the ends of the span as shown in the Figure 3.3 which shows the sine wave and the approximation of sine-wave, 4_{th} and 8_{th} order derivatives.

3.3 Application

3.3.1 Testing on Stationary Signals

To test the accuracy of the proposed method, initially, it was tested to determine the frequency from the data generated by a single sine wave of the form

$$g_1[n] = 1.0 \sin(\omega_c n T) \tag{3.14}$$

where the continous frequency

$$f_c = \frac{\omega_c}{2\pi}$$

then the Nyquist frequency

$$f_{nr} = 2 f_c \tag{3.15}$$

and the sampling frequency is chosen to be

$$f_s = \eta f_{nr} \tag{3.16}$$

η is chosen to be 4. Sampling time interval

$$T = \frac{1}{f_s} = \frac{\pi}{4\omega_c} \tag{3.17}$$

for this case the discrete frequency

$$f_d = f_c T \leq \frac{1}{2} \tag{3.18}$$

$$f_d = \frac{\omega_c}{40} \tag{3.19}$$

say if $\omega_c = 5$ then $T = \frac{2\pi}{40}$ and $f_d = 0.125$.

After this two different sine waves were superimposed together and the same was also tried with the sine waves of three different frequencies. A lot many such sets and

combinations were tried. The data generated in the case of three frequencies was from the signal $y[n]$, where

$$y[n] = g_1[n] + g_2[n] + g_3[n] \tag{3.20}$$

where,

$$g_1[n] = 1.0 \sin[\omega_1 n]$$

$$g_2[n] = 1.0 \sin[\omega_2 n]$$

$$g_3[n] = 1.0 \sin[\omega_3 n]$$

where ω_1, ω_2 and ω_3 are three different signal angular frequencies and f_1, f_2 and f_3 are the corresponding signal frequencies.

3.3.2 Testing on Non-Stationary Signals

After testing the procedure on these simple cases, real time non-stationary ECG and EEG signals were attempted.

ECG signal is a difficult signal to analyse as it contains very slow variations alongwith suddenly arising fast variations. But this difficulty was overcome by changing the span (the number of samples) accordingly, as has already been discussed.

EEG is a highly non-stationary signal. EEG is the electrical activity of the brain, monitored by scalp electrodes. The EEG signal level is typically 10 to 200 microvolts and occupies a frequency range from near DC to approximately 30 Hz. The classification of EEG is based upon its frequency contents. There are four fundamental rythms seen in an adult human brain [10], [11].

1) **Alpha Rythm.** The usual range of alpha rythm frequency is from 8 to 13 Hz. The normal amplitude of alpha rythm tends to be between 10 to 150 microvolts. Alpha

rythm occurs during relaxed wakefulness when there is little information input to the eyes, particularly when they are closed.

2) **Beta Rythm.** This rythm has a frequency range of 12 Hz and above. These have the voltages of the order of 10 microvolts and below.

3)**Theta Rythm.** This rythm has a frequency range of 4 to 7 Hz.

4) **Delta Rythm.** This rythm has a frequency range of less than 4 Hz. These are waves of large amplitude, often over 100 microvolts.

The signal data on which the frequency estimation was carried out was of Beta rythm type. It was taken from EEG done on a male of 25 years of age.

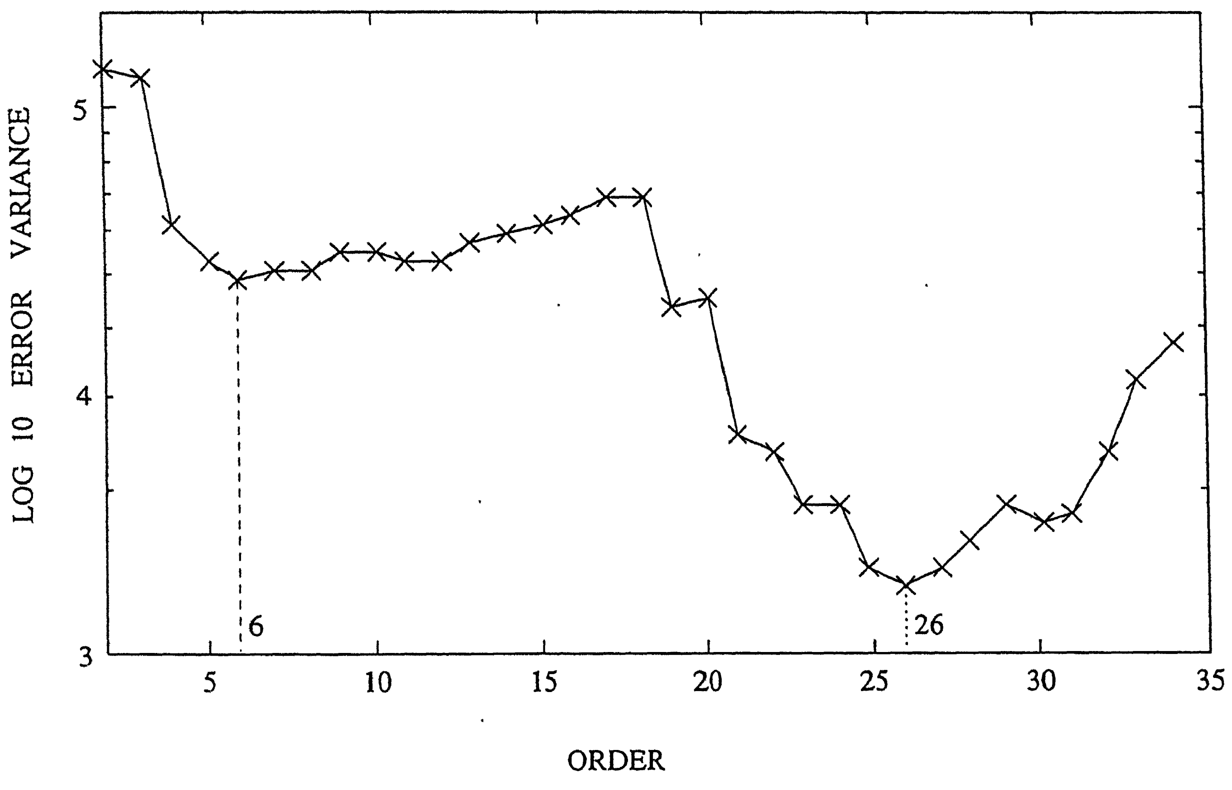

Figure 3.1: Error Variance Plot for EEG Signal (35 points).

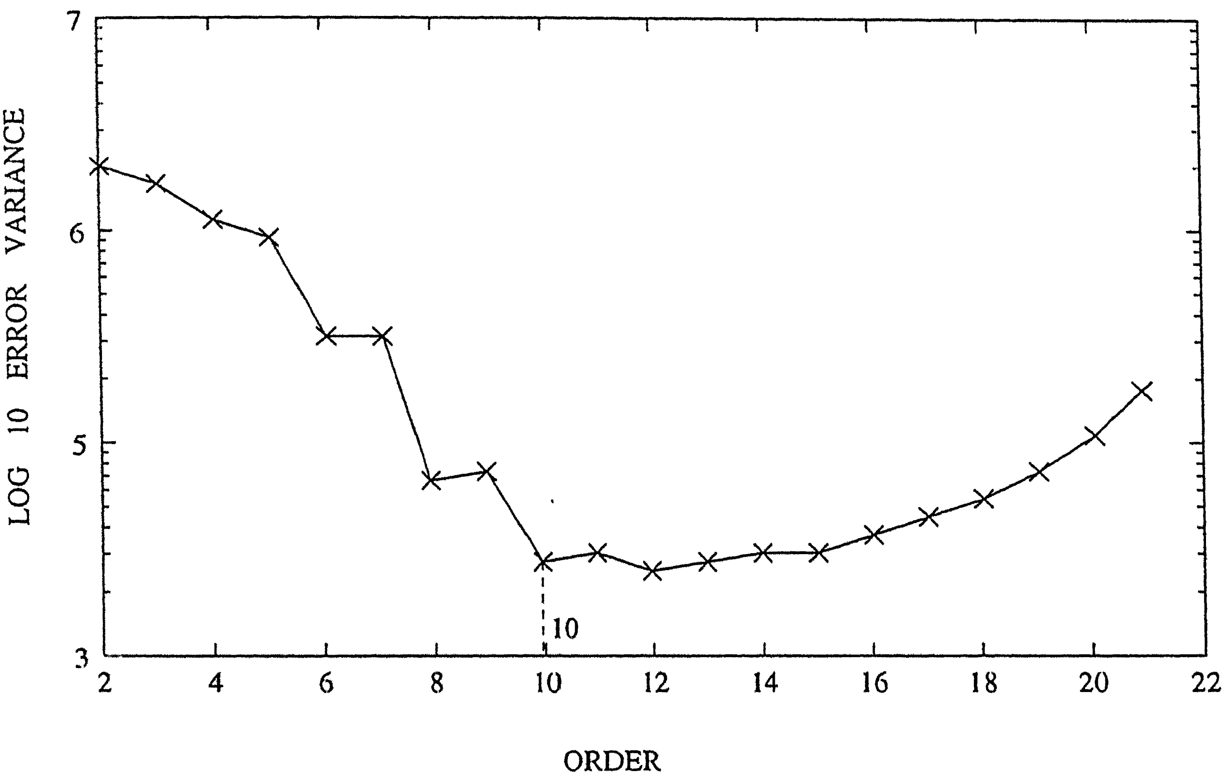

Figure 3.2: Error Variance Plot for ECG Signal (22 points).

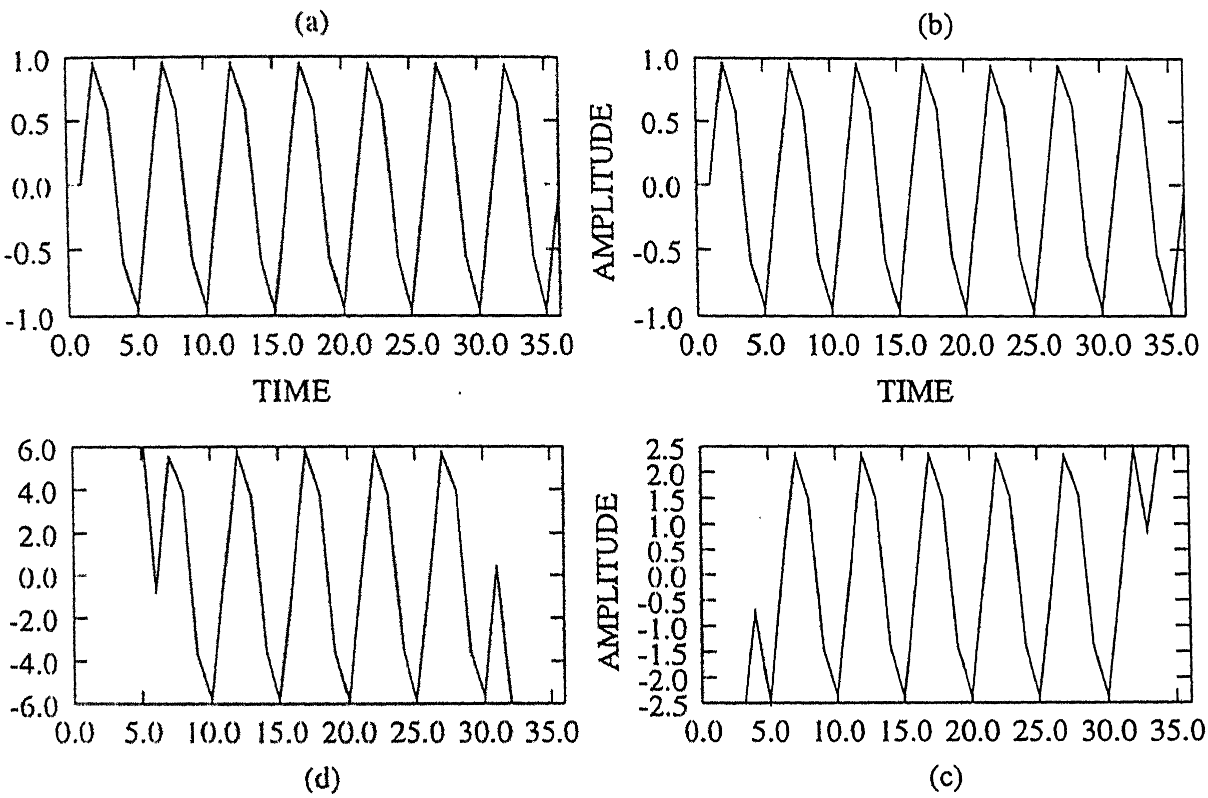

Figure 3.3: (a) Sine-wave. (b) Approximation of Sine-wave. (c) Approximation of 4_{th} order derivative and (d) Approximation of 8_{th} order derivative through orthogonal polynomials.

Chapter 4

Results and Analysis

The frequency has been estimated for single sine wave and mixture of several number of sine waves. Figure 4.1 shows the frequency estimate for the case where angular frequency $\omega_1 = 10$. A thirty point span is considered to estimate the frequency of the signal. Figure 4.2 shows the frequency estimate for the case where angular frequencies $\omega_1 = 8$, and $\omega_2 = 3$. A thirty point span has been considered to estimate the frequency. Figure 4.3 shows the frequency estimate for the superposition of three sine waves with the angular frequencies as $\omega_1 = 6, \omega_2 = 3$ and $\omega_3 = 2$. A twenty point span has been considered to estimate the frequency. The results are in error at the ends of the span as explained in chapter 3.

4.1 Simulation Results for ECG Signal

Typical ECG signal taken for the frequency estimation is shown in the Figure 4.4. Its frequency estimate with respect to the time axis is shown in the Figure 4.5 . Alongwith these results the frequency estimates got for the same data from auto-regressive method has also been given. Time span i.e., the number of sample points for these has been chosen as 40 sample points.

4.2 Simulation Results for EEG Signal

Typical EEG signal of a 25 year male taken for the frequency estimation is shown in Figure 4.6. It has continuously changing frequencies and is highly non-stationary in nature. Its frequency estimate with respect to time axis is shown in Figure 4.7. Alongwith these results the frequency estimates got for the same data from Auto-Regressive method has also been given.

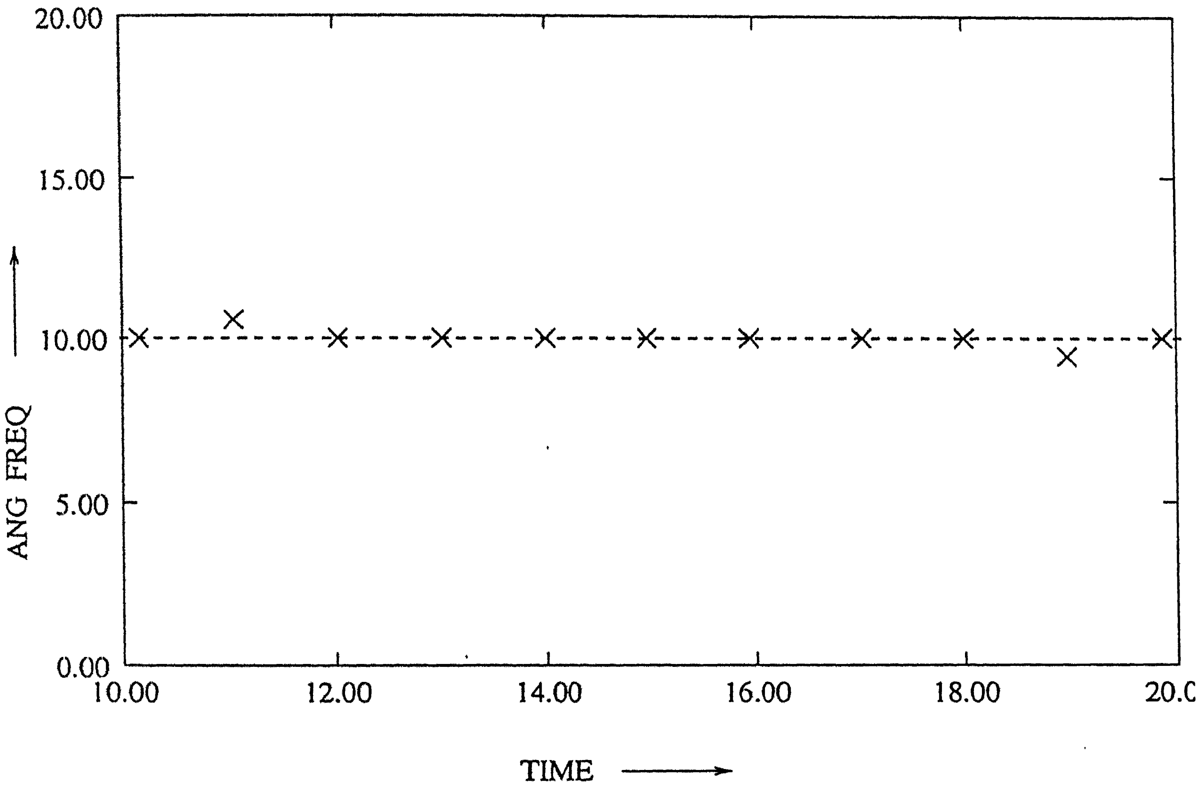

Figure 4.1: Sine Wave with angular frequency $\omega_1 = 10$.

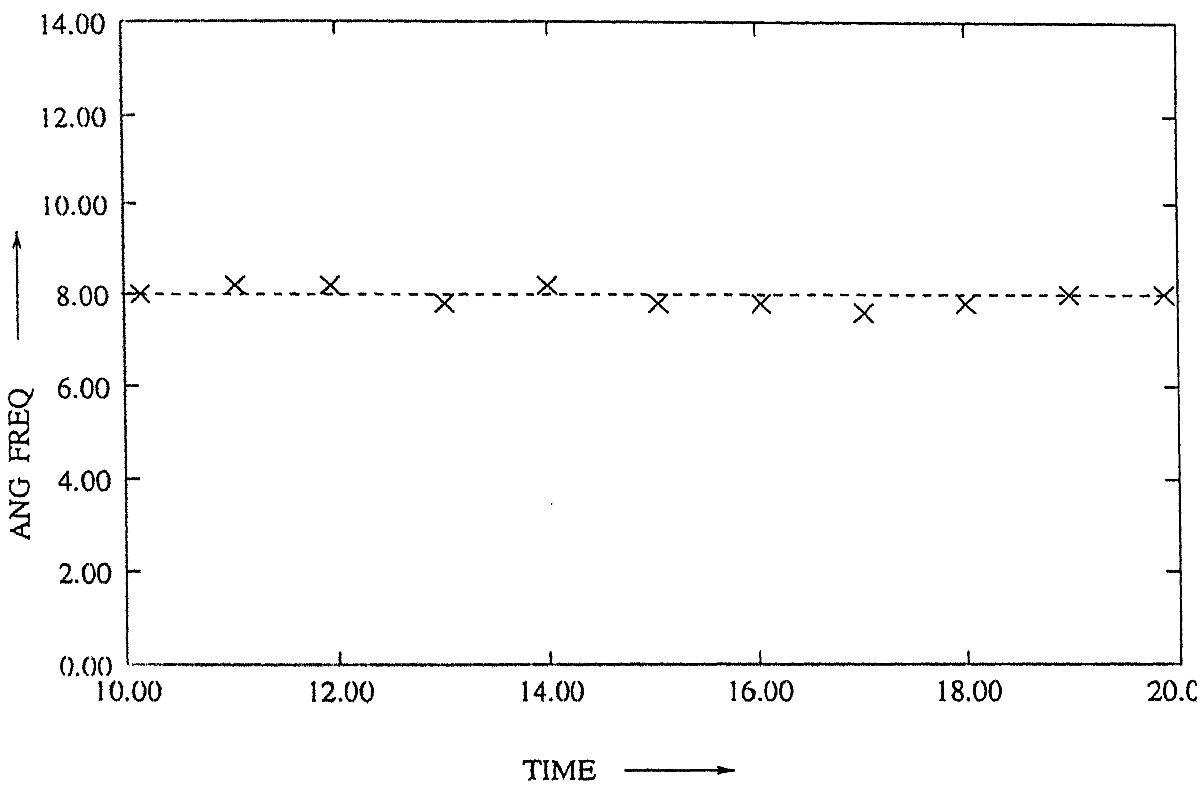

Figure 4.2: Two Sine Waves with angular frequencies $\omega_1 = 8$, and $\omega_2 = 3$.

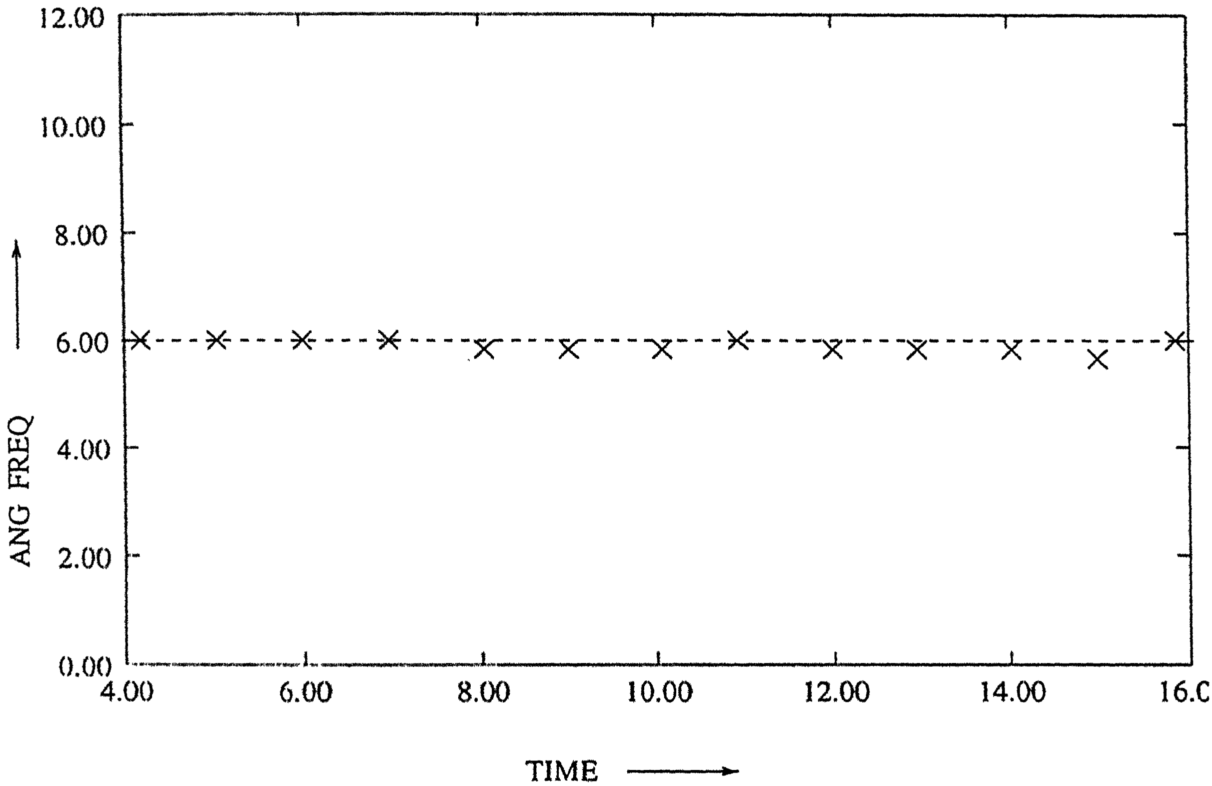

Figure 4.3: Three Sine Waves with angular frequencies $\omega_1 = 6, \omega_2 = 3$ and $\omega_3 = 2$.

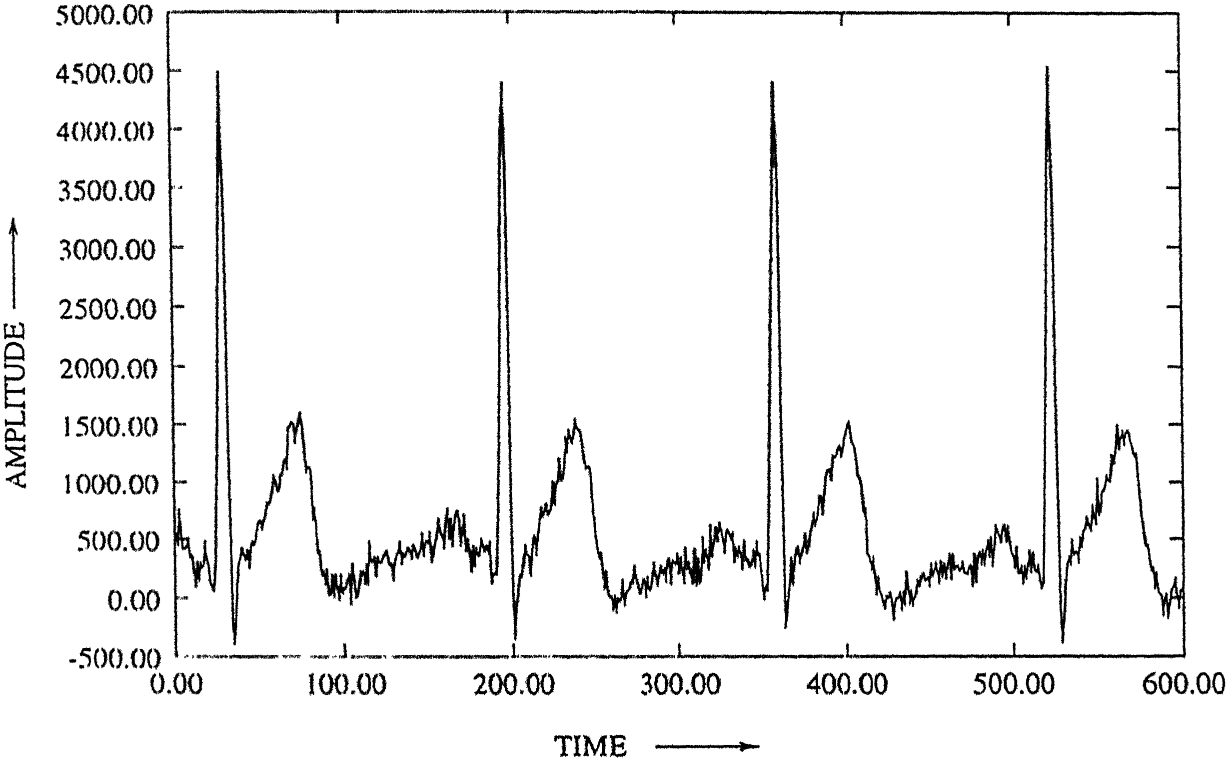

Figure 4.4: ECG Signal.

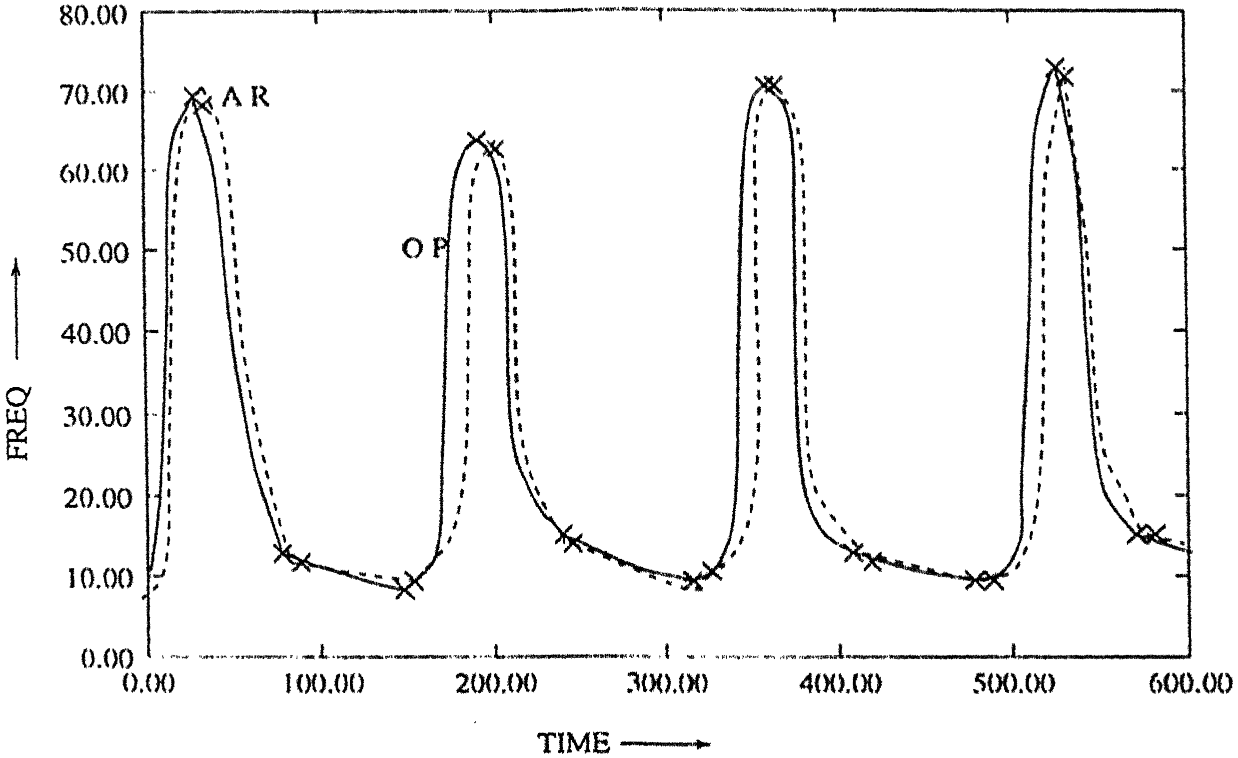

Figure 4.5: Frequency Estimation for ECG Signal by Orthogonal Polynomial and Auto-Regressive Method.

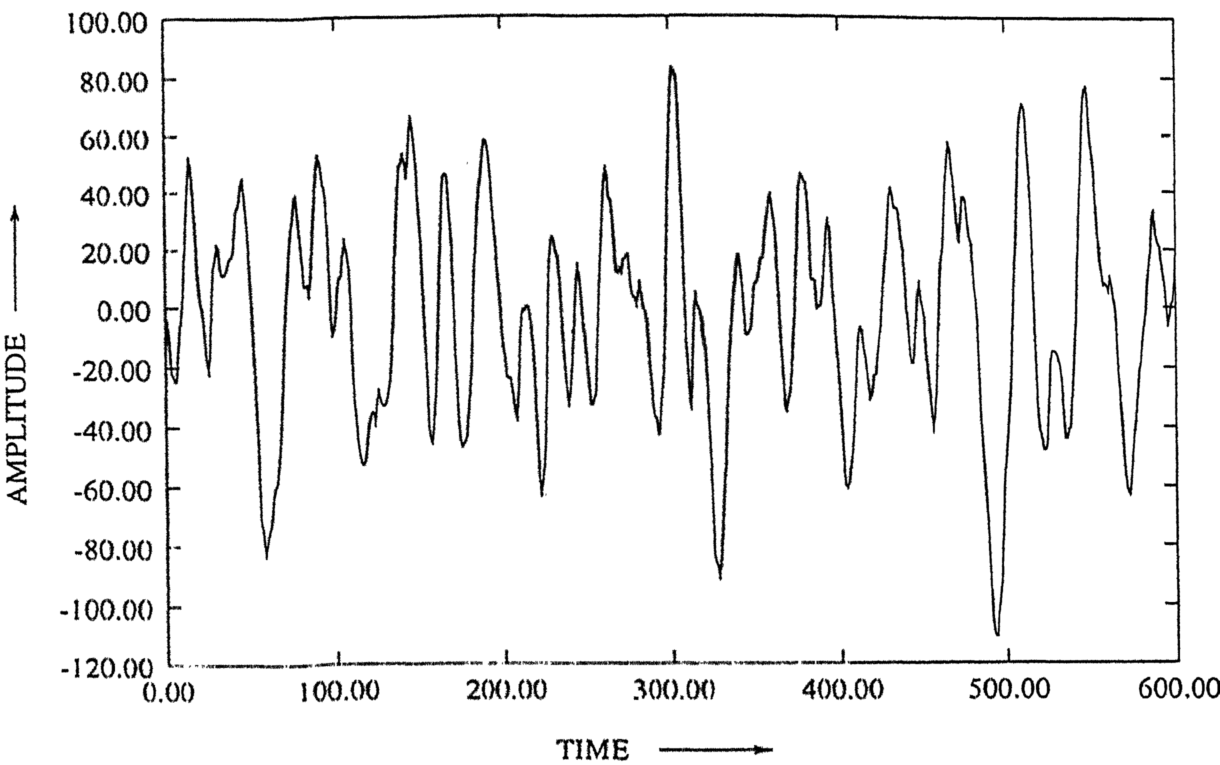

Figure 4.6: EEG Signal.

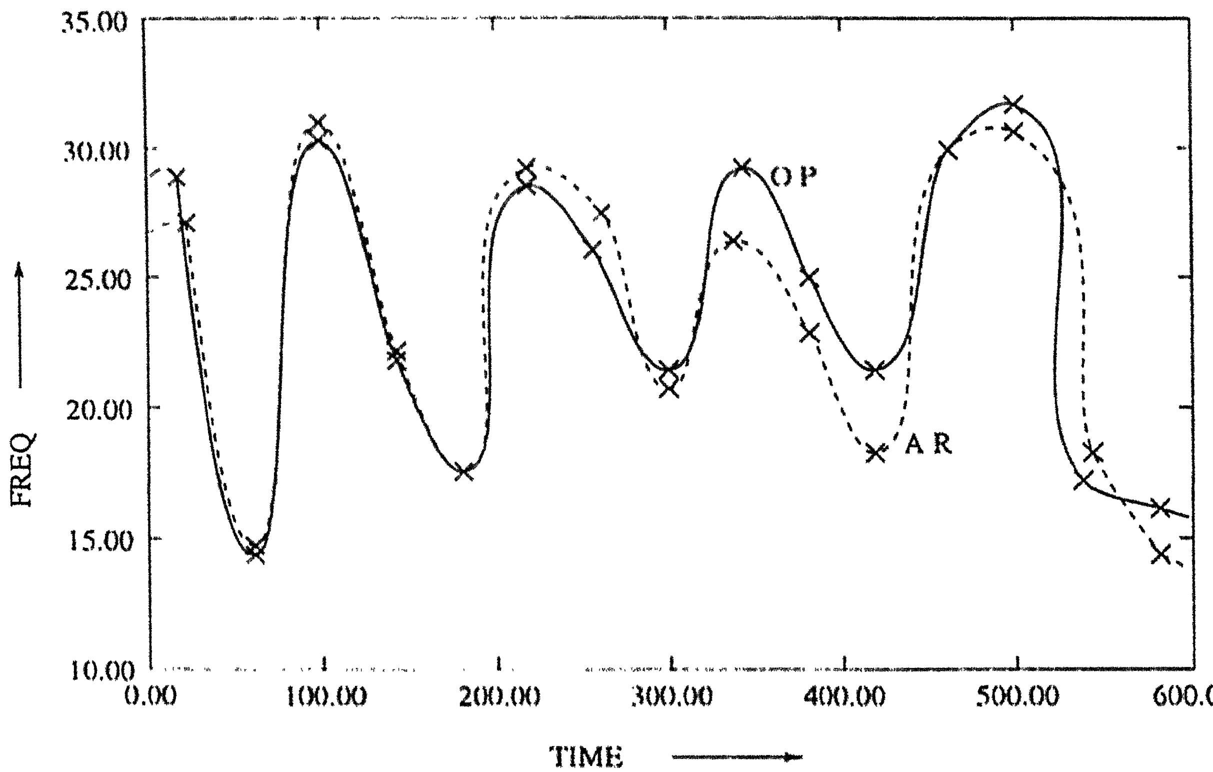

Figure 4.7: Frequency Estimation for EEG Signal by Orthogonal Polynomial and Aut
Regressive Method.

Chapter 5

Conclusions

The results for ECG and EEG signals has also been verified by using auto-regressive method and the estimation through orthogonal polynomial approximation. Both the results match quite closely as shown in chapter 4. Though more accurate, the auto-regressive method is much more computationally intensive as compared to the estimation of local bandwidth using orthogonal polynomial approximation. The estimation of local bandwidth through the use of orthogonal polynomial approximation is quite easy in its concept and is very easy to implement. It is also not computationally intensive. So it has many uses and can be used for the on-line applications.

5.1 Future Work

Finding the order of approximation is relatively a complexer job, as has been discussed in chapter 3. A more comprehensive and elaborate algorithm can be developed to estimate the order of approximation by studying a number of different signals.
This method of local bandwidth estimation can be applied to get the non-uniform sampling and the advantage gained thereby can be studied.
The hardware implementation of this method can be carried out and can be used in non-uniform sampling scheme to change the sampling interval.

Appendix A

Algorithm to Obtain Orthogonal Polynomial Approximation and its Derivatives

The algorithm to get the Orthogonal Polynomial Approximation is obtained by using equations (2.2), (2.3),(2.6),(2.4) and (2.5).

- Input data

$$(x_i, \overline{f}_i), i = 1, \ldots, m$$

$$n$$

- Algorithm

for $i = 1, \ldots, m$

do

$$p_{-1}(x_i) \leftarrow 0$$

$$p_0(x_i) \leftarrow 1$$

$$p_0^1(x_i) \leftarrow 0$$

$$p_1^1(x_i) \leftarrow 1$$

$$p_0^n(x_i) \leftarrow 0 \qquad n \geq 2$$

$$p_1^n(x_i) \leftarrow 0 \qquad n \geq 2$$

$$y_n(x_i) \leftarrow 0$$

endfor

$$N_0 \leftarrow m$$

$$b_0 \leftarrow 0$$

for $j = 0, 1, \ldots, n$

do

$$\mu_j \leftarrow \sum_{i=1}^{m} \overline{f}_i p_j(x_i)$$

$$A_j \leftarrow \frac{\mu_j}{N_j}$$

for $i = 1, \ldots, m$

do

$$y_n(x_i) \leftarrow y_n(x_i) + A_j p_j(x_i)$$

$$y_n^n(x_i) \leftarrow y_n^n(x_i) + A_j p_j^n(x_i)$$

endfor

if $j = n$ then stop

$$a_{j+1} \leftarrow \frac{\sum_{i=1}^{m} x_i [p_j(x_i)]^2}{N_j}$$

for $i = 1, \ldots, m$

do

$$p_{j+1}(x) \leftarrow (x_i - a_{j+1})p_j(x_i) - b_j p_{j-1}(x_i)$$

$$p^n_{j+1}(x) \leftarrow np^{n-1}_j + (x_i - a_{j+1})p^n_j(x_i) - b_j p^n_{j-1}(x_i)$$

endfor

$$N_j \leftarrow \sum_{i=1}^{m}[p_j(x_i)]^2$$

$$b_{j+1} \leftarrow \frac{N_{j+1}}{N_j}$$

endfor

- Output

$A_j, j = 0, \ldots, n$

$a_j, j = 1, \ldots, n$

$b_j, j = 1, \ldots, (n-1)$

$y_n(x_i), i = 1, \ldots, m$

$y^n_n(x_i), i = 1, \ldots, m$

- To find the minimum error-variance, equation (2.9) is used. The algorithm for finding the minimum error-variance is:

for $k = 0, \ldots, n$

do

$$\sigma^2_k \leftarrow \frac{\sum_{i=1}^{m}[\overline{f}_i - \sum_{j=0}^{k} A_j p_j(x_i)]^2}{m - k - 1}$$

endfor

Choose k for which σ_k^2 is minimum. For this value of k compute $y_n(x_i), i = 1, \ldots, m$. This gives the least squares approximation of $\overline{f}_i$ at data points $\{x_i\}, i = 1, \ldots, m$.

Though in practice the algorithm for finding the order of approximation is not as simple as this, as is shown in chapter 3. More complex logic has been implemented to suit the particular case.

References

[1] James J. Clark, Matthews R. Palmer and D. Lawrence, "A Transformation Method for the Reconstrution of Functions from Non-Uniformly Spaced Samples ", IEEE Transactions of Acoustics, Speech and Signal Processing, Vol. ASSP-33, No.4, Oct. 1985.

[2] K. Horiuchi, "Sampling Principle for Continuous Signals with Time Varying Bands ", Inform. Control, Vol. 13, pp 53-61, 1968.

[3] R. Basu, "On the Strategy of Non-Uniform Sampling ", M.Tech Thesis, May, 1992.

[4] P. Sircar and T. K. Sarkar, "System Identification from Non-Uniformly Spaced Measurements ", J. Signal Processing, Vol.14, No.3, pp 253-268, April, 1988.

[5] Ralston, "A First Course in Numerical Analysis", *McGraw-Hill Book Company, Kogakusha Company Ltd.*, Tokyo, 1965.

[6] Jaroslav Pachner, "Handbook of Numerical Analysis Applications", *McGraw-Hill Book Company*, New York, 1984.

[7] R. V. Southwell, "Relaxation Methods in Theoretical Physics", *Oxford, The Clarendon Press*, 1956.

[8] Rice J. Richard, "Numerical Methods, Software and Analysis", *McGraw-Hill Book Company*, New York, 1983.

[9] M. G. Salvadori and M. L. Baron, "Numerical Methods in Engineering", *Prentice Hall*, Englewood Cliff, N. J., 1961.

[10] K. H. E. Kroemer and H. J. Kroemer, "Engineering Physiology Bases of Human Factors/Ergonomics", *Van Nostrand Reinhold*, New York, 1990.

[11] B. K. Anand and S. K. Manchanda, "Textbook of Physiology", *McGraw-Hill Publishing Company Ltd*, New Delhi, 1976.